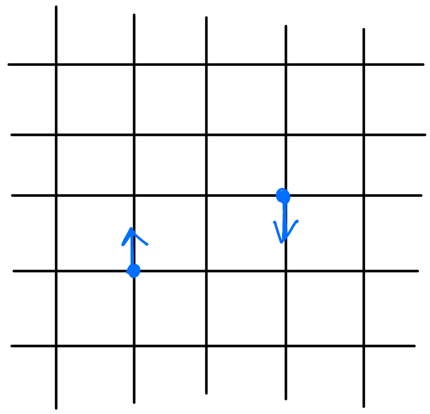
Last time : Ising model



Generating function:

$$Z[H_i] = \sum_{\{s_i\}} \exp\left(s_i K_{ij} s_j + H_i s_i\right)$$

with $K_{ij} = \beta J_{ij}$, $\beta = (k_B T)^{-1}$, $H_i = \beta h_i$

Using the identity (Homework 1)

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \exp\left(-\frac{1}{4} x_i V_{ij}^{-1} x_j + s_i x_i\right)$$

$$= \text{Const.} \times \exp\left(s_i V_{ij} s_j\right)$$

where V is any symmetric positive definite matrix, we find

$$Z[H_i] = \sum_{\{s_i\}} \exp\left(s_i K_{ij} s_j + H_i s_i\right)$$

$$= \sum_{\{s_i\}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\frac{1}{4} \phi_i K_{ij}^{-1} \phi_j + (\phi_i + H_i) s_i\right]$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\frac{1}{4}(\phi_i - H_i) K_{ij}^{-1} (\phi_j - H_j)\right] \times \sum_{\{s_i\}} \exp(\phi_i s_i)$$

Now we have

$$\sum_{\{s_i\}} \exp(\phi_i s_i) = \prod_i (2\cosh\phi_i) = \text{Const.} \times \exp\left[\sum_i \ln(\cosh\phi_i)\right]$$

Perform the linear trf. :

$$\psi_i = \frac{1}{2} K_{ij}^{-1} \phi_j \qquad (*)$$

then

$$Z[H_i] \sim \exp\left(-\frac{1}{4} H_i K_{ij}^{-1} H_j\right)$$

$$(1) \qquad \times \int \mathcal{D}\psi \exp\left(-\psi_i K_{ij} \psi_j + H_i \psi_i + \sum_i \ln[\cosh(2K_{ij}\psi_j)]\right)$$

$\longrightarrow$ except for the trivial prefactor, the external field H plays the role of the source for the generating functional

The free part

Transform to momentum space

$$\psi_i \equiv \psi(r_i) = \frac{1}{\sqrt{N}} \sum_{\vec{k}} \exp(-i\vec{k}\cdot\vec{r}_i)\, \psi(\vec{k})$$

Similarly,

$$K_{ij} = K(\vec{r}_i - \vec{r}_j) = \frac{1}{N} \sum_{\vec{k}} \exp\left[-i\vec{k}\cdot(\vec{r}_i - \vec{r}_j)\right] K(\vec{k})$$

$\rightarrow$ equation (*) becomes $\quad \phi(\vec{k}) = 2K(\vec{k})\,\psi(\vec{k})$

two contributions to bilinear part:

- $\psi_i K_{ij} \psi_j = \sum_{\vec{k}} K(\vec{k})\,\psi(\vec{k})\,\psi(-\vec{k})$
- $\ln\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots$

$$\rightarrow \; 2\sum_i \left(K_{ij}\psi_j\right)^2 = 2\sum_{\vec{k}} K(\vec{k})\psi(\vec{k})K(-\vec{k})\psi(-\vec{k})$$

$\rightarrow$ the free part of $\mathcal{L}$ in (1) is:

$$\int \mathcal{L}_0\, dx = \sum_{\vec{k}} \left[K(\vec{k}) - 2|K(\vec{k})|^2\right]\psi(\vec{k})\psi(-\vec{k}) \qquad (2)$$

Next, we expand the coefficient to second order in $|\vec{k}|$ (justification provided later):

$$K(\vec{k}) = K_0\left(1 - \rho^2 k^2\right) + \mathcal{O}(k^4)$$

Using $K(\vec{k}) = \sum_{\vec{R}} K(\vec{R}) \exp(i\vec{k}\cdot\vec{R})$, we find

- $K_0 = \sum_{\vec{R}} K(\vec{R}) = \gamma\beta J_0$

  $\uparrow$ # of nearest neighbors for each spin

- $K_0\rho^2 k^2 = \frac{1}{2}\sum_{\vec{R}} K(\vec{R})(\vec{k}\cdot\vec{R})^2 \sim K_0 a^2 k^2$

which implies that $\rho \sim a$ – the lattice constant.

Inserting back into (2), we find

$$\int dx\, \mathcal{L}_0 = \sum_{\vec{k}} \psi(\vec{k})\psi(-\vec{k}) K_0 \left[(1-2K_0) + (4K_0 - 1)\rho^2 k^2 \right]$$

Recall $K_0 = \gamma \beta J_0 \longrightarrow$ as T decreases, K_0 increases

$\longrightarrow$ at some point $(1-2K_0) = 0$

say at $T_0 = 2\gamma J_0$

$\longrightarrow$ field amplitude with $k=0$ becomes unstable (large fluctuations)

as it doesn't show up in probability distribution! $\longrightarrow$ Phase transition!

$\longrightarrow$ for finite $\vec{k}$, amplitude $\psi(\vec{k})$ is stable

Now we rewrite $\mathcal{L}_0$ as an expansion in the neighborhood of T_0:

$$1 - 2K_0 = \frac{T - T_0}{T_0} + \mathcal{O}(T-T_0)^2$$

$$4K_0 - 1 = 1 + \mathcal{O}(T - T_0)$$

$$K_0 = \frac{1}{2} + \mathcal{O}(T - T_0)$$

and $\displaystyle \int dx\, \mathcal{L}_0 = \frac{1}{2} \sum_{\vec{k}} \left(\frac{T - T_0}{T_0} + \rho^2 k^2 \right) \psi(\vec{k})\psi(-\vec{k})$

Finally, set

$$\phi = \rho \psi$$

$$\mu^2 = \frac{1}{\rho^2} \frac{T - T_0}{T_0}$$

Then,

$$\int dx\, \mathcal{L}_0 = \frac{1}{2} \sum_{\vec{k}} (\vec{k}^2 + \mu^2) \phi(\vec{k}) \phi(-\vec{k})$$

Performing the Fourier-transform

$$\bar{\phi}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \exp(-i\vec{k}\cdot\vec{r}) \phi(\vec{k}),$$

with $|\vec{k}| < \Lambda$, we get $(\Lambda \sim \frac{1}{a})$

$$\int dx\, \mathcal{L}_0 = \frac{1}{2} \int dx \left[(\nabla \bar{\phi})^2 + \mu^2 \bar{\phi}^2 \right]$$

Note $\bar{\phi}(\vec{r}) = \left(\frac{N}{V}\right)^{1/2} \phi(r_i) = a^{-d/2} \phi(\vec{r}_i)$

$$\rightarrow \quad [\bar{\phi}(\vec{r})] = L^{1-d/2}, \qquad [\mu^2] = L^{-2}$$

Green Function:

$$G_0(\vec{k}) = \langle \phi(\vec{k}) \phi(-\vec{k}) \rangle_0$$

$$= \frac{\int \mathcal{D}\phi\, \phi(\vec{k}) \phi(-\vec{k}) \exp\left[-\sum_{\vec{k}} \frac{1}{2} (k^2 + \mu^2) \phi(\vec{k}) \phi(-\vec{k}) \right]}{\int \mathcal{D}\phi \exp(-\int \mathcal{L}_0)}$$

Note $G_0(\vec{r}) = \langle \phi(\vec{r}) \phi(0) \rangle_0 = \frac{1}{V} \sum_{\vec{k}} \exp(-i\vec{k}\cdot\vec{r}) G_0(\vec{k})$

when $V \to \infty$, we get

$$G_0(\vec{r}) = \int \frac{dk}{(2\pi)^d} \exp(-i\vec{k}\cdot\vec{r}) G_0(\vec{k}) \qquad (**)$$

Adding the term $\sum_{\vec{k}} \phi(\vec{k}) h(-\vec{k})$ to $\mathcal{L}_0$, we can write

$$G_0(\vec{k}) = \frac{1}{Z^0[h]} \left. \frac{\partial^2 Z^0[h]}{\partial h(\vec{k}) \partial h(-\vec{k})} \right|_{h=0} \quad (3)$$

Using

$$Z^0[h] = \int \mathcal{D}\phi \exp\left(-\sum_{\vec{k}} \left[\frac{1}{2}(k^2+\mu^2)\phi(\vec{k})\phi(-\vec{k}) + \phi(\vec{k})h(-\vec{k})\right]\right)$$

$$= \left[\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\sum_{\vec{k}} \phi(\vec{k})(k^2+\mu^2)\phi(-\vec{k})\right)\right]$$

$$\times \exp\left(\frac{1}{2}\sum_{\vec{k}} h(\vec{k})(k^2+\mu^2)^{-1} h(-\vec{k})\right) \quad (4)$$

where we shifted the integration variable

$$\phi(\vec{k}) \longmapsto \phi(\vec{k}) - (k^2+\mu^2)^{-1} h(\vec{k})$$

Using eq. (3), we get

$$G_0(\vec{k}) = (k^2+\mu^2)^{-1}$$

Recall that the susceptibility was

$$\chi = \sum_{\vec{R}} G_0(\vec{R}) \overset{(**)}{=} G_0(\vec{k}=0) = \mu^{-2}$$

It diverges when $\mu^2 \longrightarrow 0$

$\longrightarrow$ signals Phase transition!

§ 4.1 Renormalization Group

Recall one-loop renormalization for Non-Abelian gauge theories:

$$\Gamma_R^{(4)} \doteq (1+L_A)\,\underbrace{\Gamma^{(4)}}_{-\frac{1}{4}\int d^4x\, F_{\gamma\mu\nu} F_\gamma^{\mu\nu}} \tag{1}$$

where $L_A = \frac{-g^2}{2\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right)\ln\left(\frac{\Lambda}{\mu}\right) + O(g^4)$

and $\Gamma^{(N)}$ denotes the terms arising from loop integrals with N external fields

$Z_A^{1/2}(g,\mu,\Lambda) \equiv (1+L_A)^{1/2}$ is called the "field renormalization constant"

(recall $A^R_{\alpha\mu} \equiv (1+L_A)^{1/2} A_{\alpha\mu}$)

Eq. (1) can be generalized to ϕ^4-theory (exercise):

$$\Gamma_R^{(E)}(k_i; g(k_1), k_1) = Z_\phi^{E/2}\,\Gamma^{(E)}(k_i;\lambda,\Lambda) \tag{2}$$

external momenta

with $Z_\phi = 1 + g z_1 + g^2 z_2 + \cdots$

where the z_i are functions of the cutoff Λ.

Equation (2) has been derived in the massless case when imposing the following normalization conditions:

$$\Gamma_R^{(2)}(0\,;g) = 0$$

$$\frac{\partial}{\partial k^2}\Gamma_R^{(2)}(k\,;g)\Big|_{k^2=\kappa^2} = 1 \qquad (3)$$

We have $Z_\phi = Z_\phi(g(\kappa_1), \kappa_1, \Lambda)$

Remarks!

- Right-hand side of (2) is the limit of left-hand side as $\Lambda \to \infty$, and is finite
- Z_ϕ diverges logarithmically at $d=4$ as $\Lambda \to \infty$, with finite g.

Assume now that κ_2 were used in the normalization conditions instead of κ_1

$$\to \Gamma_R^{(N)}(k_i\,;g(\kappa_1),\kappa_1) = \left[Z(\kappa_2, g_2, \kappa_1, g_1)\right]^{N/2}\Gamma_R^{(N)}(k_i\,;g(\kappa_2),\kappa_2)$$

where we have set $g_i \equiv g(\kappa_i)$ and defined

$$Z(\kappa_2, g_2, \kappa_1, g_1) = Z_\phi(g_1, \kappa_1, \Lambda)/Z_\phi(g_2, \kappa_2, \Lambda)$$

which is finite in the limit $\Lambda \to \infty$.